NEUTRAL WAVES ON THE SURFACE OF A FILM OF LIQUID FLOWING ALONG A VERTICAL WALL INTO ANOTHER LIQUID

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The film of liquid flowing down along a vertical or inclined surface into a surrounding different liquid is encountered in some forms of extraction equipment and in electric-arc fusion of metals. It is for this reason that this type of flow is of interest from the practical point of view. On the other hand, the film of liquid forming a boundary with another liquid is interesting from the theoretical point of view as an example of a flow in which the flow in the external medium affects the surface waves in the thin liquid layer.

The inflow of a film of liquid along an inclined or vertical wall into another liquid can be separated into three flow regions (Fig. 1). In the region of the inlet section I, the flow is primarily determined by the conditions at the inlet and represents a near-wall stream. In the region of the initial section II, the conditions at the inlet no longer affect the flow, and the boundary layer developing in the external liquid has not yet reached the wall of the channel. In region III, the flow is a steady state flow which does not depend on x.

The linear stability of the flow under examination in region III has been investigated in a number of papers [1-4], where it is shown that in region III, as in the case of a free film of liquid [5], the stability of the flow is related to the development of waves on the interface. The critical Reynolds number Re_* equals zero for flow in the vertical channel and differs from zero in the inclined flow. The stability of the flow in the region of the initial section II was not investigated in these papers. However, as shown in [6, 7], the boundary layer between two flows affects the stability of the flow. This paper is concerned with the investigation of the linear stability of the flow in the region of the initial section II. We examine the case when the liquids are immiscible, the flow is isothermal, and the external liquid is stationary at infinity.

As is often done in analyzing the stability of boundary-layer flows [6-8], we shall assume that the unperturbed flow does not depend on the longitudinal coordinate x and the boundary layer in the external liquid has a finite thickness δ_p. Experiments [9] show that in region II, when the quantity δ_p exceeds the thickness of the film of liquid h_p, the quantity h_p is nearly independent of x. For this reason, in the case $\delta_p > h_p$ the unperturbed velocity profile can be approximated by a Nusselt profile [10] in the liquid film and by a Polhauzen profile [8], corresponding to a zero-gradient flow past a plate, in the boundary layer. In dimensionless form, where the average thickness of the liquid film h_0 and the average flow velocity in it U_0 are the length and velocity scales, the unperturbed velocity profile, which satisfies the conditions of continuity of velocity and tangential stresses of the interface, has the form

$$\begin{gathered} U_1(y)=3\left[\left(1+\frac{\tau}{6+\tau}\right)y-\left(1+\frac{3\tau}{6+\tau}\right)\frac{y^2}{2}\right],\quad 0\leqslant y\leqslant 1, \\ U_2(y)=\frac{9}{6+\tau}\left(1+\frac{y-1}{\delta}\right)\left(\frac{1+\delta-y}{\delta}\right)^3,\quad 1\leqslant y\leqslant 1+\delta, \\ U_3(y)=0,\quad 1+\delta\leqslant y<\infty,\quad \tau=3\mu/\delta. \end{gathered} \tag{1}$$

In this case, for the average thickness of the liquid film we have

$$h_0=\sqrt[3]{\frac{3\,\mathrm{Re}\,\nu_1^2}{g(1-\rho)}\left(1+\frac{3\tau}{6+\tau}\right)}, \tag{2}$$

where $\rho = \rho_2/\rho_1$, $\mu = \mu_2/\mu_1$ are the relative density and dynamic viscosity, respectively (in denoting physical properties, the index 1 refers to the liquid forming the film and the index 2 refers to the external liquid); $Re = h_0U_0/\nu_1$ is the Reynolds number based on the flow rate.

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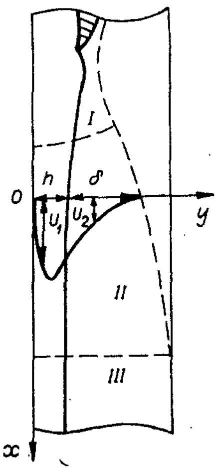

Fig. 1

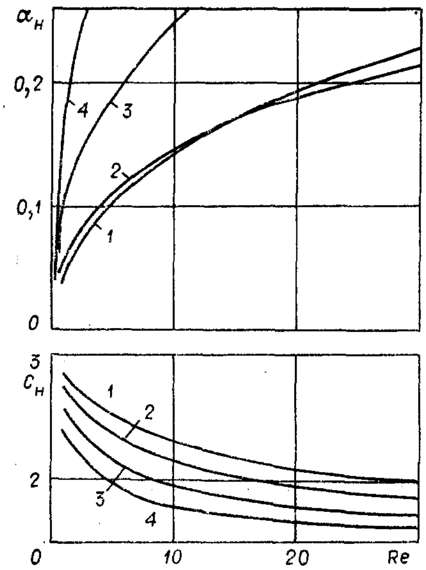

Fig. 2

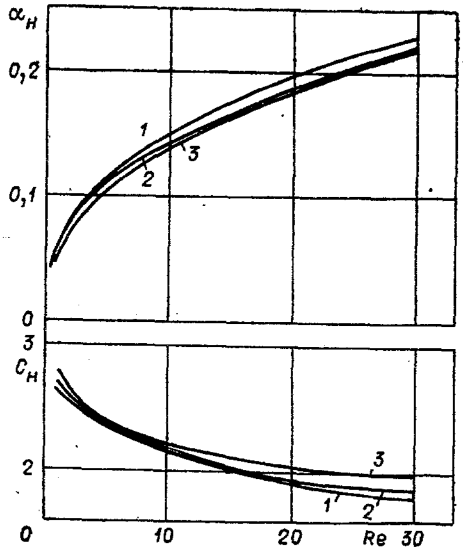

Fig. 3

In investigating the stability of the flow under examination relative to infinitesimal perturbations, we shall seek all perturbed quantities in the form

$$A(x,\ y,\ t) = A_0(y)\exp\,[i\alpha(x - c_* t)], \tag{3}$$

where α, $c_* = c + i\eta$ are the wave number and the complex phase velocity, respectively.

Neglecting the nonlinear terms in the equations of motion of the liquid and using (3), we obtain the Orr–Sommerfeld equations for the amplitudes of the stream functions $\varphi_i(y)$ of the perturbed flow:

$$\varphi_1^{\mathrm{IV}} - [i\alpha\,\mathrm{Re}\,(U_1 - c_*) + 2\alpha^2]\,\varphi_1'' + [i\alpha\,\mathrm{Re}\,(\alpha^2(U_1 - c_*) + U_1'') + \alpha^4]\,\varphi_1 = 0, \quad 0 \leqslant y \leqslant 1; \tag{4}$$

$$\varphi_2^{\mathrm{IV}} - \left[i\alpha\frac{\mathrm{Re}}{\nu}(U_2 - c_*) + 2\alpha^2\right]\varphi_2'' + \left[i\alpha\frac{\mathrm{Re}}{\nu}(\alpha^2(U_2 - c_*) + U_2'') + \alpha^4\right]\varphi_2 = 0, \quad 1 \leqslant y \leqslant 1 + \delta; \tag{5}$$

$$\varphi_3^{\mathrm{IV}} - \left[-i\alpha\frac{\mathrm{Re}}{\nu}c_* + 2\alpha^2\right]\varphi_3'' + \left[-i\alpha\frac{\mathrm{Re}}{\nu}\alpha^2 c_* + \alpha^4\right]\varphi_3 = 0, \quad 1 + \delta \leqslant y < \infty. \tag{6}$$

The boundary conditions of attachment on the wall, continuity of the velocity and stresses on the surface of the liquid film and on the external boundary of the boundary layer, as well as damping of the perturbations at infinity can be written, using the kinematic condition at the interface, in the form

$$y = 0:\ \varphi_1 = 0,\ \varphi_1' = 0; \tag{7}$$

$$\begin{gathered} y = 1:\ U_1'\varphi_1 + (c_* - U_1)\,\varphi_1' = U_2'\varphi_2 + (c_* - U_2)\,\varphi_2', \quad \varphi_1 = \varphi_2, \\ U_1''\varphi_1 + (c_* - U_1)\left(\varphi_1'' + \alpha^2\varphi_1\right) = \mu\left[U_2''\varphi_2 + (c_* - U_2)\left(\varphi_2'' + \alpha^2\varphi_2\right)\right], \\ (c_* - U_1)\left[(c_* - U_1)\,\varphi_1' + U_1'\varphi_1\right] - \frac{i}{\alpha\,\mathrm{Re}}\left[\left(\varphi_1''' - 3\alpha^2\varphi_1'\right)(c_* - U_1) - \right. \\ \left. - 2\alpha^2\varphi_1 U_1'\right] - \alpha^2\varphi_1\mathrm{We} = \rho\,(c_* - U_2)\left[(c_* - U_2)\,\varphi_2' + U_2'\varphi_2\right] - \\ - \frac{i\mu}{\alpha\,\mathrm{Re}}\left[\left(\varphi_2''' - 3\alpha^2\varphi_2'\right)(c_* - U_2) - 2\alpha^2\varphi_2 U_2'\right]; \end{gathered} \tag{8}$$

$$y = 1 + \delta:\ \varphi_2 = \varphi_3,\quad \varphi_2' = \varphi_3',\quad \varphi_2'' = \varphi_3'',\quad \varphi_2''' = \varphi_3'''; \tag{9}$$

$$y \to \infty:\ \varphi_3,\ \varphi_3' \to 0, \tag{10}$$

where $\nu = \nu_2/\nu_1$ is the relative kinematic viscosity; $\mathrm{We} = \sigma/h_0 U_0 \rho_1$ is Weber's number. According to (2), we have

$$\mathrm{We}=\sqrt[3]{\frac{3\mathrm{Fi}}{\mathrm{Re}^5}\left(1+\frac{3\tau}{6+\tau}\right)}, \tag{11}$$

where $\mathrm{Fi} = \sigma^3/(\rho_1^3\nu_1^4 g(1-\rho))$.

Thus the linear analysis of the stability of the flow under examination reduces to the problem of finding the eigenvalues of the Orr–Sommerfeld equations (4)-(6) with the boundary conditions (7)-(10).

In this work, this problem was solved numerically with the help of the determinant method [11]. In application to the flow under examination, this method consists of the following.

Let φ_{ij} $(i = 1,\ldots,3,\ j = 1,\ 2)$ be linearly independent solutions of Eqs. (4)-(6), where φ_{11}, φ_{12} and φ_{31}, φ_{32} satisfy conditions (7), (10), respectively, while φ_{21}, φ_{22} are the continuation of the functions φ_{31}, φ_{32} from the region $y \geqslant 1+\delta$ into the region $1 \leqslant y \leqslant 1+\delta$ with the help of conditions (9). Then the solution of the problem has the form

$$\begin{aligned}
\varphi_1 &= A_{11}\varphi_{11} + A_{12}\varphi_{12}, \quad 0 \leqslant y \leqslant 1,\\
\varphi_2 &= A_{21}\varphi_{21} + A_{22}\varphi_{22}, \quad 1 \leqslant y \leqslant 1+\delta,\\
\varphi_3 &= A_{31}\varphi_{31} + A_{32}\varphi_{32}, \quad 1+\delta \leqslant y \leqslant \infty.
\end{aligned} \tag{12}$$

Substituting (12) into (8), we obtain a homogeneous system of linear algebraic equations for the coefficients A_{11}, A_{12}, A_{21}, A_{22}, for which the condition for the existence of nontrivial solution can be written in the form

$$F(\alpha, c_*, \mathrm{Re}, \mathrm{We}, \delta, \rho, \mu, D_{ij}) = 0, \tag{13}$$

where

$$\begin{aligned}
D_{i1} &= \begin{vmatrix}\varphi_{i1} & \varphi_{i2}\\ \varphi'_{i1} & \varphi'_{i2}\end{vmatrix}, \quad D_{i2} = \begin{vmatrix}\varphi'_{i1} & \varphi'_{i2}\\ \varphi''_{i1} & \varphi''_{i2}\end{vmatrix}, \quad D_{i3} = \begin{vmatrix}\varphi''_{i1} & \varphi''_{i2}\\ \varphi'''_{i1} & \varphi'''_{i2}\end{vmatrix},\\
D_{i4} &= \begin{vmatrix}\varphi'_{i1} & \varphi'_{i2}\\ \varphi'''_{i1} & \varphi'''_{i2}\end{vmatrix}, \quad D_{i5} = \begin{vmatrix}\varphi_{i1} & \varphi_{i2}\\ \varphi'''_{i1} & \varphi'''_{i2}\end{vmatrix}, \quad D_{i6} = \begin{vmatrix}\varphi_{i1} & \varphi_{i2}\\ \varphi''_{i1} & \varphi''_{i2}\end{vmatrix}.
\end{aligned} \tag{14}$$

The expression for the function F in (13) is not presented due to its cumbersomeness. It can be shown that the functions D_{ij} satisfy the equations

$$\begin{aligned}
D'_{i1} = D_{i6},\quad D'_{i2} = D_{i4},\quad D'_{i3} = B_iD_{i6},\quad D'_{i4} = D_{i3} + A_iD_{i2} + B_iD_{i1},\\
D'_{i5} = D_{i4} + A_iD_{i6},\quad D'_{i6} = D_{i2} + D_{i5},
\end{aligned} \tag{15}$$

where $A_1 = i\alpha\,\mathrm{Re}\,(U_1 - c_*) + 2\alpha^2$; $B_1 = i\alpha\,\mathrm{Re}[(U_1 - c_*)\alpha^2 + U_1''] + \alpha^4$; $A_2 = i\alpha\frac{\mathrm{Re}}{\nu}(U_2 - c_*) + 2\alpha^2$; $B_2 = i\alpha\frac{\mathrm{Re}}{\nu}\times [(U_2 - c_*)\alpha^2 + U_2''] + \alpha^4$.

The boundary conditions for the functions D_{ij}, by virtue of the homogeneity of the boundary-value problems (4)-(10), have the form

$$y = 0:\ D_{11} = D_{12} = D_{14} = D_{15} = D_{16} = 0,\ D_{13} = 1; \tag{16}$$

$$y = 1+\delta:\ D_{2j} = D_{3j},\ j = 1,\ldots,6. \tag{17}$$

The functions D_{3j} in (16) can be written out explicitly, since from Eq. (6) we have

$$\varphi_{31} = e^{-\alpha y}, \varphi_{32} = e^{-\lambda y},\ \lambda = \sqrt{\alpha^2 - i\alpha\mathrm{Re}c/\nu}. \tag{18}$$

Finally, using (14), (17), and (18), we obtain for D_{2j}

$$\begin{aligned}
y = 1+\delta:\ D_{21} &= (\alpha - \lambda)\exp[-(\alpha+\lambda)(1+\delta)],\\
D_{22} &= \alpha\lambda(\alpha-\lambda)\exp[-(\alpha+\lambda)(1+\delta)],\ D_{23} = \alpha^2\lambda^2(\alpha-\lambda)\\
&\times \exp[-(\alpha+\lambda)(1+\delta)],\ D_{24} = \alpha\lambda(\lambda^2-\alpha^2)\exp[-(\alpha+\lambda)\\
&\times(1+\delta)],\ D_{25} = (\alpha^3-\lambda^3)\exp[-(\alpha+\lambda)(1+\delta)],\ D_{26}\\
&= (\lambda^2-\alpha^2)\exp[-(\alpha+\lambda)(1+\delta)].
\end{aligned} \tag{19}$$

To suppress the rapidly growing solutions of Eqs. (15) we introduce the new functions $z_{ij} = D_{ij}/D_{i3}$ and Eqs. (15) with the boundary conditions (16), (19), by virtue of the fact that the system (15) has the integral $D_{i6}D_{i4} = D_{i1}D_{i3} + D_{i2}D_{i5}$, reduce to the system of equations

$$\begin{aligned} &z'_{i2} = z_{i4} - B_i z_{i6} z_{i2}, \qquad i = \begin{cases} 1, & 0 \leqslant y \leqslant 1, \\ 2, & 1 \leqslant y \leqslant 1 + \delta, \end{cases} \\ &z'_{i4} = A_i z_{i2} - B_i z_{i2} z_{i5} + 1, \\ &z'_{i5} = z_{i4} + A_i z_{i6} - B_i z_{i6} z_{i5}, \quad z'_{i6} = z_{i2} + z_{i5} - B_i z_{i6} \end{aligned} \tag{20}$$

with the boundary conditions

$$\begin{gathered} y = 0\colon\ z_{12} = z_{14} = z_{15} = z_{16} = 0,\ y = 1 + \delta\colon\ z_{22} = \\ = 1/(\alpha\lambda),\ z_{24} = -(\alpha + \lambda)/(\alpha\lambda),\ z_{25} = (\alpha^3 - \lambda^3)/(\alpha^2\lambda^2(\alpha - \lambda)), \\ z_{26} = -(\alpha + \lambda)/(\alpha^2 + \lambda^2). \end{gathered} \tag{21}$$

Equation (13) is written in the form

$$y = 1\colon\ F_1(\alpha,\ c,\ \mathrm{Re},\ \mathrm{We},\ \delta,\ \rho,\ \mu,\ z_{ij}) = 0. \tag{22}$$

Thus the problem of finding the eigenvalues of (4)-(10) reduces to solving the algebraic equation (22), in which the functions z_{ij} are solutions of the Cauchy problem for Eqs. (20) with the boundary conditions (21).

In this work we calculated the wave numbers $\alpha_H = f(\mathrm{Re})$ and phase velocities $c_H = \varphi(\mathrm{Re})$, corresponding to neutral perturbations ($\eta = 0$). Both in the case of a free liquid film [5] and with a combined flow of two liquids in a channel (in the region of the steady-state flow III) [1], the instability of the flow for Re is due to the presence of the interface. The phase velocity of waves on this interface equals approximately 1.5-3 and decreases with increasing Re and decreasing Fi. In the region of the initial section II, the experimentally observed undamped waves on the interface also have a velocity $c \approx 2$, which decreases with increasing Re [9]. It is natural to suppose that the stability of the flow under examination is also related to surface waves in region II. For this reason, in this work, we investigated the mode of oscillations corresponding to these waves. The region of instability ($\eta > 0$) for this mode of oscillations in the α – Re plane is bounded by a curve emanating from the origin of coordinates and by the axis $\alpha = 0$, on which, as follows from the solution of the system (4)-(10), for $\alpha = 0$ the velocity of waves equals $c = 3(1 + \tau/(6 + \tau))$. The second boundary of the region of instability was determined numerically. The calculations were performed for Re = 1-30, Fi = $5\cdot10^6$–$5\cdot10^{10}$, μ = 0.125-8, ρ = 0.4-0.9999, $\delta \geqslant 4$. The results of the calculation are presented in Figs. 2-4. Based on these results, we note the following.

An increase in the value of ρ from 0.4 to 0.9999 with constant Fi leads to some decrease of the velocity of neutral waves and does not have a significant effect on the neutral curves $\alpha_H = f(\mathrm{Re})$ [Fig. 2, $\mu = 1$, $\delta = 8$, curve 1 corresponds to $\rho = 0.4$, Fi = $5\cdot10^{10}$ 2, $\rho = 0.9999$, Fi = $5\cdot10^{10}$, 3, $\rho = 0.9999$, Fi = $5\cdot10^8$, 4, $\rho = 0.9999$, Fi = $5\cdot10^6$]. An increase in Fi leads to an increase in the velocity of neutral waves and a narrowing of the region of instability [since Fi $\sim \sigma^3/(\rho_1^3\nu_1^4(1 - \rho))$], this parameter can change by several orders of magnitude when σ, ρ_1, ν_1 change by several factors and ρ changes by an amount of the order of one.

An increase in the thickness of the boundary layer (Fig. 3, Fi = $5\cdot10^{10}$, $\rho = 0.9999$, $\mu = 1$, $\delta = 4;\ 8;\ \infty$ for curves 1-3) leads to a narrowing of the region of instability for small Re. For Re $\geqslant 30$, as δ increases, the region of instability is smallest for some finite, depending on Fi, ρ, μ, Re, value of δ. The velocity of neutral waves decreases for small Re and increases for large Re with increasing δ.

An increase in μ for finite values of δ (Fig. 4a, $\delta = 8$, Fi = $5\cdot10^{10}$, $\rho = 0.9999$, $\mu = 0.125;\ 1;\ 8$ for curves 1-3) increases the width of the region of instability for small Re. For large values of Re the region of instability is smallest for some finite, depending on Fi, δ, ρ, Re, value of μ. The velocity of neutral waves increases with increasing μ for small Re and is smallest for some finite, depending on Fi, δ, ρ, Re, value of μ for large Re. The relative dynamic viscosity μ affects the stability of the flow, on the one hand, as a parameter in the equations of the perturbed flow and, on the other, through the unperturbed velocity profile and, as shown in [3], through the magnitude of the discontinuity of the derivative dU/dy at $y = 1$. When $\delta \to \infty$, the flow in the external medium becomes uniform, and the unperturbed velocity profile does not depend on μ, while $dU/dy = 0$ in the limit $\delta \to \infty$.

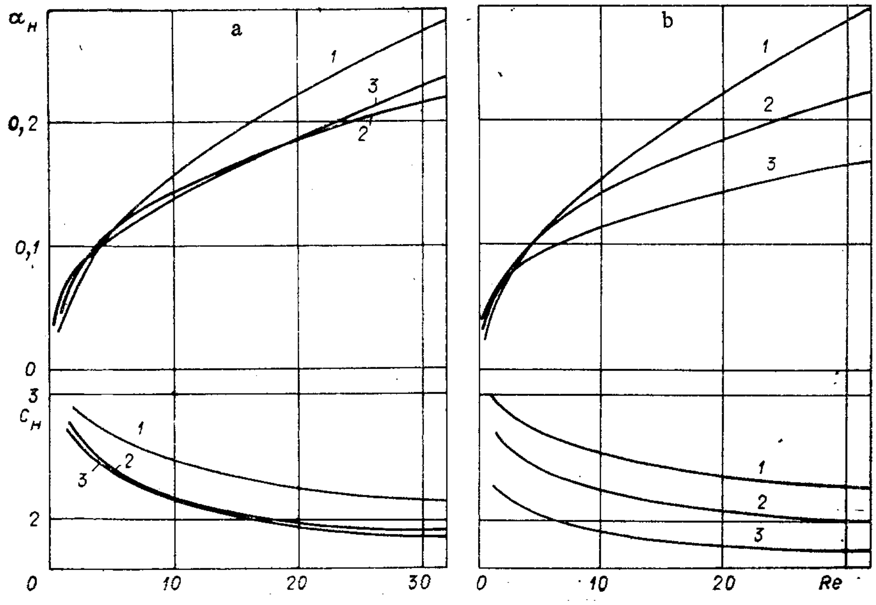

Fig. 4

Therefore, in this case, μ affects the stability of the flow only as the parameter in the equations for the unperturbed flow. As μ increases (Fig. 4b, $\delta = \infty$), the region of instability widens for small Re and narrows for large Re, and the velocity of the waves decreases.

Thus, when a film of liquid flows along a vertical wall into another liquid which is stationary at infinity, undamped waves exist at the interface for any Re. For Re $\lesssim$ 30 the characteristics of these waves for fixed Re are determined primarily by the parameter Fi and, to a lesser extent, by μ and δ.

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